Bull. Math. Soc. Sci. Math. Roumanie
Tome 52(100) No. 3, 2009, 271–279

A Weitzenböck formula for a closed Riemannian manifold with two orthogonal complementary distributions

by
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To Professor S. Ianuş on the occasion of his 70th Birthday

Abstract

In the setting of a closed Riemannian manifold endowed with a smooth, not necessarily integrable distribution, we prove that a Weitzenböck type formula, given in [14] for the particular case of a Riemannian foliation, can be extended in our more general case.

Key Words: smooth distribution; Riemannian foliation; Weitzenböck formula
2000 Mathematics Subject Classification: Primary 58A30, Secondary 53C12, 14F17.

1 Introduction

In the general setting of a closed Riemannian manifold (M, g), the geometric properties of Dirac and Laplacian operators are known to have a crucial role (see e.g., [5],[4]). During the last period of time, the relevant featured of the above mentioned operators has been carried out also in the special frame of the so called Riemannian foliations, first defined in [18]. To point out just several aspects in this research field we remind that for the so called *basic* Laplacians which act on the de Rham complex of *basic* differential forms, the study concerning Hodge theory was mainly done in [8] and [10], the construction of heat kernels, asymptotic expansions and some elements of spectral geometry was stated in [6], [7], [11], [13] and [21], while the use of a Weitzenböck-like formula in [15] allows the authors to establish classical vanishing results in the case of basic cohomology.

Other fundamental differential operators that one can define on a Riemannian foliation are the *leafwise* Laplacian operator (see e.g., [3]), the *transversal* Dirac operator [9], or the *transversal* Laplacian on differential forms (if one extends the setting of basic differential forms associated to a Riemannian foliation to the general de Rham complex defined on the closed manifold M [2]). In both *leafwise* and *transversal* cases Weitzenböck-like formulas can be derived (see [3],[14],[19],[20]), and we are lead to useful estimates for the Laplacians [3], or even vanishing results for a Riemannian foliation endowed with a transversally almost complex structure [14].

Very recently, in the attempt to study the equivariant index of a K-theory class of some transversally elliptic operators, I. Prokhorenkov and K. Richardson ([12]) have taken into account a new class of Dirac-type operators associated to a couple of smooth, not necessarily integrable distributions $Q, L \subset TM$, with $L = Q^{\perp}$, defined on a closed Riemannian manifold. This is in fact a natural extension of the case of a Riemannian foliation with normal bundle. These operators are transversally elliptic and essentially self-adjoint with respect to the inner products associated to the closed Riemannian manifold [12].

The main purpose of this paper is to show that a Weitzenböck type formula, valid in the case of Riemannian foliations and presented in [14], can be extended in the larger setting considered in [12].

Throughout this paper we are focussed on the action of a *transversal* Dirac-type operator when it applies on differential forms associated to the *transversal* distribution Q, even thought a similar approach might be considered in the frame of transversal Clifford modules (see [14], [9], [12] etc.). In Section 2 one introduces the operators $d_{\Gamma(\bigwedge Q^*)}$, $\delta_{\Gamma(\bigwedge Q^*)}$ —which correspond to the exterior derivative and coderivative operators in our setting, and one points out the existing relation with the Dirac-type operators defined in [12]. Using this and following [14], in the last Section we work out a Weitzenböck type formula which extends a previous one presented in [14].

2 Canonical transversal differential operators

In what follows let us consider a closed Riemannian manifold (M, g), and assume that $Q, L \subset TM$ are two smooth distributions on M such that $L = Q^{\perp}$. Q will be called *transversal* or *vertical* distribution, while L *horizontal* distribution. We do not assume that Q or L are integrable (see [12]). As a consequence, the tangent and the cotangent vector bundles associated to M split as follows

$$\begin{aligned} TM &= Q \oplus L, \\ TM^* &= Q^* \oplus L^*. \end{aligned}$$

The canonical transversal and horizontal projection operator will be denoted by π_L and π_Q respectively. We consider the complex $\Gamma\left(\bigwedge Q^*\right) := \left\{\Gamma\left(\bigwedge^a Q^*\right) | a = 1, .., q := dim\, Q\right\}$ of transversal differential forms defined on the manifold M.

Throughout this paper we will consider local vector fields $\{f_a, e_i\}$ defined on a neighborhood of an arbitrary point $x \in M$, so that they determine an orthonormal basis at any point where they are defined, $\{f_a\}$ spanning the distribution Q and $\{e_i\}$ spanning the distribution L. Let us consider also the dual coframe $\left\{\alpha^a, \beta^i\right\}$ for $\{f_a, e_i\}$.

We consider in the following the local vector fields U and V on M and denote by U^Q, V^Q the transverse components, and by U^L, V^L the horizontal components. First of all, we define the following two tensor fields $\mathcal{A}$ and $\mathcal{T}$:

$$\begin{aligned} \mathcal{T}_U V &:= \pi_L \nabla_{U^L} V^Q + \pi_Q \nabla_{U^L} V^L, \\ \mathcal{A}_U V &:= \pi_L \nabla_{U^Q} V^Q + \pi_Q \nabla_{U^Q} V^L, \end{aligned}$$

where ∇ is the Levi-Civita connection associated to g. We canonically extend the above operators on differential forms.

Remark 1. *In the special case of a Riemannian submersion, the above tensor fields are just the classical Gray-O'Neill tensor fields* (*see* [16]).

Let us define the metric connection ∇^Q by:

$$\nabla^Q_U V := \pi_Q \nabla_U V^Q,$$

and the corresponding one for the horizontal distribution:

$$\nabla^L_U V := \pi_L \nabla_U V^L.$$

As a consequence, the Levi-Civita connection splits as follows:

$$\nabla_U = \nabla_U^Q + \nabla_U^L + \mathcal{A}_{UQ} + \mathcal{T}_{UL}. \tag{1}$$

On can naturally extend ∇^Q to a metric connection on the complex $\Gamma\left(\bigwedge Q^*\right)$ associated to the transversal distribution Q.

Now, using the local orthonormal frames $\{f_a, e_i\}$ and $\left\{\alpha^a, \beta^i\right\}$, we define the following first order differential operators on $\Gamma\left(\bigwedge Q^*\right)$, which can be regarded as corresponding to exterior derivative and coderivative operators in our frame, that can be conceived as an extension of the considerations of [1]:

$$\begin{aligned} d_{\Gamma(\bigwedge Q^*)} \quad &: = \sum_a \alpha^a \wedge \nabla_{f_a}^Q, \\ \delta_{\Gamma(\bigwedge Q^*)} \quad &: = -\sum_a i_{f_a} \nabla_{f_a}^Q + i_\tau, \end{aligned} \tag{2}$$

where we set $\tau := \sum_i \mathcal{T}_{e_i} e_i$ to be the mean curvature vector field associated to the distribution L (see [12], [22]), while $\alpha^a \wedge \cdot$ and $i_{f_a} \cdot$ represent the *exterior* and *interior* product on differential forms, respectively. We also define the *mean curvature form* k which is subject to the condition $k(U) = g(\tau, U)$, for any vector field U.

Remark 2. *It is easy to see that* $k(\tau) = \|\tau\|^2$.

Let us consider the pointwise inner product $(\cdot \mid \cdot)$ on M determined by the metric tensor field g. We remind the definition of the divergence of an arbitrary smooth vector field X in our local frame $\{f_b, e_i\}$, that is (see e.g., [5])

$$div\,(X) = \left(\sum_a \nabla_{f_a}^Q X \mid f_a\right) + \left(\sum_i \nabla_{e_i} X \mid e_i\right). \tag{3}$$

Also, as in [14], let us notice that $\left(\nabla_X^Q\right)^* = -\nabla_X^Q - div\,(X)$, where the adjoint operator is considered with respect to the inner product on the closed Riemannian manifold (M, g).

Proposition 1. *The above defined operators* $d_{\Gamma(\bigwedge Q^*)}$ *and* $\delta_{\Gamma(\bigwedge Q^*)}$ *are adjoint with respect to the canonical inner product obtained by integrating on the closed Riemannian manifold.*

Proof: For any α_1, $\alpha_2 \in \Gamma\left(\bigwedge Q^*\right)$ and $\alpha \in \Gamma\left(\bigwedge^1 Q^*\right)$, we denote by X_{α_1,α_2} the global vector field on M defined by the relation $\alpha(X_{\alpha_1,\alpha_2}) = (\alpha \wedge \alpha_1 \mid \alpha_2)$. First of all one gets

$$\begin{aligned} \left(d_{\Gamma(\bigwedge Q^*)}\alpha_1 \mid \alpha_2\right) &= \sum_a \left(\nabla_{f_a}^Q \left(\alpha^a \wedge \alpha\right)_1 \mid \alpha_2\right) - \sum_a \left(\nabla_{f_a}^Q \alpha^a \wedge \alpha_1 \mid \alpha_2\right) \\ &= \sum_a f_a \left(\alpha^a \wedge \alpha_1 \mid \alpha_2\right) - \sum_a \left(\nabla_{f_a}^Q \alpha^a \wedge \alpha_1 \mid \alpha_2\right) \\ &\quad - \sum_a \left(\alpha^a \wedge \alpha_1 \mid \nabla_{f_a}^Q \alpha_2\right) \\ &= \sum_a \alpha^a \left(\nabla_{f_a}^Q X_{\alpha_1,\alpha_2}\right) - \sum_a \left(\alpha^a \wedge \alpha_1 \mid \nabla_{f_a}^Q \alpha_2\right) \\ &= \sum_{a,b} \alpha^a \left(\left(\nabla_{f_a}^Q X_{\alpha_1,\alpha_2} \mid f_b\right) f_b\right) - \sum_a \left(\alpha^a \wedge \alpha_1 \mid \nabla_{f_a}^Q \alpha_2\right) \\ &= \sum_a \left(\nabla_{f_a}^Q X_{\alpha_1,\alpha_2} \mid f_a\right) - \sum_a \left(\alpha^a \wedge \alpha_1 \mid \nabla_{f_a}^Q \alpha_2\right). \end{aligned} \tag{4}$$

Now we take into consideration the right hand side of the equality

$$\left(\alpha_1 \mid \delta_{\Gamma(\bigwedge Q^*)}\alpha_2\right) = \left(\alpha_1 \mid \sum_a -i_{f_a}\nabla^Q_{f_a}\alpha_2\right) + (\alpha_1 \mid i_\tau\alpha_2)\,.$$

The first term can be written as follows:

$$\left(\alpha_1 \mid \sum_a -i_{f_a}\nabla^Q_{f_a}\alpha_2\right) = -\sum_a \left(\alpha^a \wedge \alpha_1 \mid \nabla^Q_{f_a}\alpha_2\right), \tag{5}$$

while the second one becomes successively:

$$\begin{aligned}(\alpha_1 \mid i_\tau\alpha_2) &= (k\wedge\alpha_1 \mid \alpha_2) \\ &= k(X_{\alpha_1,\alpha_2}) \\ &= (X_{\alpha_1,\alpha_2} \mid \tau) \\ &= \sum_i (X_{\alpha_1,\alpha_2} \mid \nabla_{e_i}e_i) \\ &= -\sum_i (\nabla_{e_i}X_{\alpha_1,\alpha_2} \mid e_i)\,.\end{aligned} \tag{6}$$

Subtracting (5) and (6) from (4), one finally gets

$$\begin{aligned}\left(d_{\Gamma(\bigwedge Q^*)}\alpha_1 \mid \alpha_2\right) - \left(\alpha_1 \mid \delta_{\Gamma(\bigwedge Q^*)}\alpha_2\right) &= \sum_a \left(\nabla^Q_{f_a}X_{\alpha_1,\alpha_2} \mid f_a\right) \\ &\quad + \sum_i (\nabla_{e_i}X_{\alpha_1,\alpha_2} \mid e_i) \\ &= div\,(X_{\alpha_1,\alpha_2}).\end{aligned}$$

If one integrates over the closed Riemannian manifold (M,g) in order to obtain the canonical inner product (see e.g., [5]), then the following relation is obtained:

$$\left\langle d_{\Gamma(\bigwedge Q^*)}\alpha_1, \alpha_2\right\rangle - \left\langle \alpha_1, \delta_{\Gamma(\bigwedge Q^*)}\alpha_2\right\rangle = \int_M div\,(X_{\alpha_1,\alpha_2})d\mu_g.$$

Provided the fact that the manifold M is closed, we obtain the desired result. □

We introduce now the Clifford multiplication $\bullet$ on forms. In accordance with [17], for any local differential 1-form θ and arbitrary local differential form ω on M, we define $\theta\bullet\omega := \theta\wedge\omega - i_{\theta^\sharp}\omega$, the local vector field $\theta^\sharp$ being determined by the relation $g\left(\theta^\sharp, U\right) = \theta(U)$, for any local vector field U.

The following statement is easy to verify (see [17]).

Proposition 2. *For any differential* 1*-forms* θ_1 *and* θ_2 *we have*

$$\theta_1\bullet\theta_2\bullet\mathrm{Id}_{\Gamma(\bigwedge Q^*)} + \theta_2\bullet\theta_1\bullet\mathrm{Id}_{\Gamma(\bigwedge Q^*)} = -2\,(\theta_1 \mid \theta_2)\,\mathrm{Id}_{\Gamma(\bigwedge Q^*)}.$$

In accordance with [12], we introduce now the transversal Dirac operator associated to the transversal distribution Q:

$$D^Q := \sum_a \alpha^a\bullet\nabla^Q_{f_a} - \frac{1}{2}k\bullet\mathrm{Id}_{\Gamma(\bigwedge Q^*)}$$

Proposition 3. *The following equality holds* (*see also* [14])

$$D^Q = d_{\Gamma(\bigwedge Q^*)} + \delta_{\Gamma(\bigwedge Q^*)} - \frac{1}{2}\left(k\wedge\mathrm{Id}_{\Gamma(\bigwedge Q^*)} + i_\tau\mathrm{Id}_{\Gamma(\bigwedge Q^*)}\right).$$

Proof: We have successively

$$\begin{aligned}
D^Q &= \sum_a \alpha^a \bullet \nabla^Q_{f_a} - \frac{1}{2} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&= \sum_a \alpha^a \bullet \nabla^Q_{f_a} - \frac{1}{2} k \wedge \mathrm{Id}_{\Gamma(\bigwedge Q^*)} + \frac{1}{2} i_\tau \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&= \sum_a \alpha^a \wedge \nabla^Q_{f_a} - \sum_a i_{f_a} \nabla^Q_{f_a} + i_\tau \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&\quad - \frac{1}{2} \left(k \wedge \mathrm{Id}_{\Gamma(\bigwedge Q^*)} + i_\tau \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \right).
\end{aligned}$$

Now, the relations (2) lead us to the desired equality. □

Remark 3. *Due to the fact that $k \wedge \mathrm{Id}_{\Gamma(\bigwedge Q^*)}$ and $i_\tau \mathrm{Id}_{\Gamma(\bigwedge Q^*)}$ are adjoint operators and using also Proposition 1, we obtain that D^Q is a self-adjoint operator* (*see* [12]).

In the classical manner (see e.g.,[4]), we define now a transversal Laplacian $\Delta^Q := \left(D^Q\right)^2$ associated to the transversal distribution Q. Note that it differs from the transversal Laplacian defined in [2] (see Proposition 3 or [14]). In the last part of this paper, following a similar approach as in [14], we work out a Weitzenböck-like formula for this transversal Laplacian Δ^Q, this way extending a previous formula stated in the particular case of a Riemannian foliation.

3 A Weitzenböck-like formula for the transversal Laplacian Δ^Q

We begin this section by studying the terms that appear in the expression of Δ^Q. Using Lemma 1 and Leibniz rule, one gets successively

$$\begin{aligned}
\Delta^Q &= \left(\sum_a \alpha^a \bullet \nabla^Q_{f_a} - \frac{1}{2} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \right) \left(\sum_b \alpha^b \bullet \nabla^Q_{f_b} - \frac{1}{2} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \right) \\
&= \left(\sum_a \alpha^a \bullet \nabla^Q_{f_a} \right) \left(\sum_b \alpha^b \bullet \nabla^Q_{f_b} \right) - \frac{1}{2} \sum_a \alpha^a \bullet \nabla^Q_{f_a} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&\quad - \frac{1}{2} \sum_a \alpha^a \bullet k \bullet \nabla^Q_{f_a} - \frac{1}{2} \sum_a k \bullet \alpha^b \bullet \nabla^Q_{f_b} \\
&\quad + \frac{1}{4} k \bullet k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)}.
\end{aligned}$$

If we consider once again the properties of the Clifford multiplication on differential forms, one obtains that

$$\begin{aligned}
-\frac{1}{2} \sum_a \alpha^a \bullet k \bullet \nabla^Q_{f_a} - \frac{1}{2} \sum_a k \bullet \alpha^a \bullet \nabla^Q_{f_a} &= \sum_a \langle \alpha^a, k \rangle \nabla^Q_{f_a} \\
&= \sum_a \left(\sum_i \nabla_{e_i} e_i \mid f_a \right) \nabla^Q_{f_a}
\end{aligned}$$

and

$$\frac{1}{4} k \bullet k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} = -\frac{1}{4} \|k\|^2 \mathrm{Id}_{\Gamma(\bigwedge Q^*)}.$$

As a consequence, we finally get

$$\begin{aligned}\left(D^Q\right)^2 &= \left(\sum_a \alpha^a \bullet \nabla^Q_{f_a}\right)^2 - \frac{1}{2}\sum_a \alpha^a \bullet \nabla^Q_{f_a} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\ &\quad + \sum_a \left(\sum_i \nabla_{e_i} e_i \mid f_a\right) \nabla^Q_{f_a} - \frac{1}{4}\|k\|^2 \mathrm{Id}_{\Gamma(\bigwedge Q^*)}.\end{aligned} \tag{7}$$

Now we are going to study the first terms of (7). As in [14], we denote $D' := \sum_a \alpha^a \bullet \nabla^Q_{f_a}$. Then we have

$$(D')^2 = \sum_{a,b} \alpha^a \bullet \left(\nabla^Q_{f_a} \alpha^b\right) \bullet \nabla^Q_{f_b} + \sum_{a,b} \alpha^a \bullet \alpha^b \bullet \nabla^Q_{f_a}\left(\nabla^Q_{f_b}\right).$$

If we express the covariant derivative $\nabla^Q_{f_a}$ on differential forms in the local frame $\{f_a\}$, then we get $\nabla^Q_{f_a}\alpha^b = -\Gamma^b_{ac}\alpha^c$. Let us notice that

$$\begin{aligned}\sum_{a,b} \alpha^a \bullet \left(\nabla^Q_{f_a}\alpha^b\right) \bullet \nabla^Q_{f_b} &= -\sum_{a,b} \alpha^a \bullet \alpha^c \bullet \nabla^Q_{\Gamma^b_{ac} f_b} \\ &= -\sum_{a,c} \alpha^a \bullet \alpha^c \bullet \nabla^Q_{\nabla^Q_{f_a} f_c} \\ &= \sum_a \nabla^Q_{\nabla^Q_{f_a} f_a} - \frac{1}{2}\sum_{a,b} \alpha^a \bullet \alpha^b \bullet \nabla^Q_{\nabla^Q_{f_a} f_b - \nabla^Q_{f_b} f_a}.\end{aligned}$$

Also, we get

$$\sum_a \nabla^Q_{\nabla^Q_{f_a} f_a} = \sum_a \left(\sum_b \nabla^Q_{f_b} f_b \mid f_a\right) \nabla^Q_{f_a}$$

and, furthermore

$$\begin{aligned}\nabla^Q_{f_a} f_b - \nabla^Q_{f_b} f_a &= [f_a, f_b] + \left(\nabla^Q_{f_a} f_b - \nabla^Q_{f_b} f_a - [f_a, f_b]^Q\right) - [f_a, f_b]^L \\ &= [f_a, f_b] + T^Q_{f_a, f_b} - [f_a, f_b]^L \\ &= [f_a, f_b] - [f_a, f_b]^L,\end{aligned}$$

where T^Q denotes the projection $\pi_Q T$ of the torsion tensor field associated to the Levi-Civita connection. As it is known it vanishes, so finally one gets

$$\begin{aligned}\sum_{a,b} \alpha^a \bullet \left(\nabla^Q_{f_a}\alpha^b\right) \bullet \nabla^Q_{f_b} &= \sum_a \left(\sum_b \nabla^Q_{f_b} f_b \mid f_a\right) \nabla^Q_{f_a} - \frac{1}{2}\sum_{a,b} \alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a, f_b]} \\ &\quad + \frac{1}{2}\sum_{a,b} \alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a, f_b]^L}.\end{aligned}$$

We also obtain that

$$\begin{aligned}\sum_{a,b} \alpha^a \bullet \alpha^b \bullet \nabla^Q_{f_a}\left(\nabla^Q_{f_b}\right) &= -\sum_a \nabla^Q_{f_a}\left(\nabla^Q_{f_a}\right) \\ &\quad + \frac{1}{2}\sum_{a,b} \alpha^a \bullet \alpha^b \bullet \left(\nabla^Q_{f_a}\left(\nabla^Q_{f_b}\right) - \nabla^Q_{f_b}\left(\nabla^Q_{f_a}\right)\right).\end{aligned}$$

As a consequence we are able to express the term $(D')^2$:

$$
\begin{aligned}
(D')^2 &= -\sum_a \nabla^Q_{f_a}\left(\nabla^Q_{f_a}\right) + \sum_a \left(\sum_b \nabla^Q_{f_b} f_b \mid f_a\right)\nabla^Q_{f_a} + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet R^Q_{f_a,f_b} \\
&+ \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a,f_b]^L},
\end{aligned}
$$

where $R^Q_{f_a,f_b} := \nabla^Q_{f_a}\left(\nabla^Q_{f_b}\right) - \nabla^Q_{f_b}\left(\nabla^Q_{f_a}\right) - \nabla^Q_{[f_a,f_b]}$ is the curvature operator canonically associated to the connection ∇^Q. Using (7), the Weitzenböck-like formula follows:

$$
\begin{aligned}
\left(D^Q\right)^2 &= -\sum_a \nabla^Q_{f_a}\left(\nabla^Q_{f_a}\right) + \sum_a \left(\sum_b \nabla^Q_{f_b} f_b + \sum_i \nabla_{e_i} e_i \mid f_a\right)\nabla^Q_{f_a} \\
&+ \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet R^Q_{f_a,f_b} + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a,f_b]^L} \\
&- \frac{1}{2}\sum_a \alpha^a \bullet \nabla^Q_{f_a} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} - \frac{1}{4}\|k\|^2 \mathrm{Id}_{\Gamma(\bigwedge Q^*)}.
\end{aligned}
$$

If one puts $X = f_a$ in (3), then

$$
\begin{aligned}
div(f_a) &= \left(\sum_b \nabla^Q_{f_b} f_a \mid f_b\right) + \left(\sum_a \nabla_{e_i} f_a \mid e_i\right) \\
&= -\left(\sum_b \nabla^Q_{f_b} f_b + \sum_a \nabla^L_{e_i} e_i \mid f_a\right).
\end{aligned}
$$

As a consequence one gets

$$
\begin{aligned}
\sum_a \nabla^{Q*}_{f_a}\left(\nabla^Q_{f_a}\right) &= \sum_a \left(-\nabla^Q_{f_a} - div(f_a)\right)\left(\nabla^Q_{f_a}\right) \qquad (8)\\
&= -\sum_a \nabla^Q_{f_a}\left(\nabla^Q_{f_a}\right) + \sum_a \left(\sum_b \nabla^Q_{f_b} f_b + \sum_i \nabla_{e_i} e_i \mid f_a\right)\nabla^Q_{f_a},
\end{aligned}
$$

and using (8), one ends up with the following theorem.

Theorem 1. *Let (M,g) be a closed Riemannian manifold with a horizontal distribution L and a transverse distribution $Q = L^\perp$, and Δ^Q the square of the transversal Dirac operator D^Q associated to Q. Then the following Weitzenböck-like formula for the Laplacian Δ^Q is true:*

$$
\begin{aligned}
\Delta^Q &= \sum_a \nabla^{Q*}_{f_a}\left(\nabla^Q_{f_a}\right) + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet R^Q_{f_a,f_b} + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a,f_b]^L} \\
&- \frac{1}{2}\sum_a \alpha^a \bullet \nabla^Q_{f_a} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \frac{1}{4}\|k\|^2 \mathrm{Id}_{\Gamma(\bigwedge Q^*)}.
\end{aligned}
$$

One can still refine the above formula observing that

$$
\begin{aligned}
\sum_a \alpha^a \bullet \nabla^Q_{f_a} k \bullet \mathrm{Id}_{\Gamma(\bigwedge Q^*)} &= d_{\Gamma(\bigwedge Q^*)} k\, \mathrm{Id}_{\Gamma(\bigwedge Q^*)} + \delta_{\Gamma(\bigwedge Q^*)} k\, \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&\quad - i_\tau k\, \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&= d_{\Gamma(\bigwedge Q^*)} k\, \mathrm{Id}_{\Gamma(\bigwedge Q^*)} + \delta_{\Gamma(\bigwedge Q^*)} k\, \mathrm{Id}_{\Gamma(\bigwedge Q^*)} \\
&\quad - \|k\|^2 \mathrm{Id}_{\Gamma(\bigwedge Q^*)},
\end{aligned}
$$

and one can restate the Weitzenböck-like formula as follows:

$$\begin{aligned}\Delta^Q &= \sum_a \nabla^{Q*}_{f_a}\left(\nabla^Q_{f_a}\right) + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet R^Q_{f_a,f_b} + \frac{1}{2}\sum_{a,b}\alpha^a \bullet \alpha^b \bullet \nabla^Q_{[f_a,f_b]^L} \\ &\quad -\frac{1}{2}d_{\Gamma(\bigwedge Q^*)}k\mathrm{Id}_{\Gamma(\bigwedge Q^*)} - \frac{1}{2}\delta_{\Gamma(\bigwedge Q^*)}k\mathrm{Id}_{\Gamma(\bigwedge Q^*)} + \frac{1}{4}\|k\|^2\,\mathrm{Id}_{\Gamma(\bigwedge Q^*)}.\end{aligned}$$

Remark 4. *This formula can be regarded as a generalization for the non-integrable case of the Weitzenböck-like formula stated in* [14, Theorem 7] *for the particular case of a Riemannian foliation endowed with a bundle-like metric so that the mean curvature form is a basic one* (*see also* [9]).

Acknowledgments

The authors thank Y.A. Kordyukov for helpful discussions.

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Received: 20.04.2009.

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